6.1 – Inner Products

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Definition: An **inner product** on a real vector space V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfied for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars k.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ (symmetry axiom)
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ (additivity axiom)
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$ (homogeneity axiom)
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$ (positivity axiom)

Definition: A real vector space with an inner product is called a **real inner product space**.

Note that an inner product is a category of operation that is performed on vectors in a vector space. Any operation that satisfies these axioms is an inner product.

Collection of objects → Set

Set together with particularly defined vector addition & scalar mult. → vector space

Vector space together with an inner product → inner product space

Examples of Inner products

- The dot product $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$ is the **Euclidean inner product** or **standard inner product** on R^n. [R^n with the Euclidean inner product is called **Euclidean n-space**.]

- If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n and $w_1, w_2, \ldots, w_n$ are positive real numbers, then the formula $\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \ldots + w_n u_n v_n$ is called the **weighted Euclidean inner product with weights** $\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}$.

- On M_{nn}, the set of $n \times n$ matrices: If $\mathbf{u} = U$ and $\mathbf{v} = V$ are matrices in the vector space M_{nn}, then the formula $\langle \mathbf{u}, \mathbf{v} \rangle = tr(U^T V)$ is the **standard inner product** on M_{nn}.

- If $\mathbf{p} = a_0 + a_1 x + \ldots + a_n x^n$ and $\mathbf{q} = b_0 + b_1 x + \ldots + b_n x^n$ are polynomials in P_n, then the **standard inner product** on P_n is $\langle \mathbf{p}, \mathbf{q} \rangle = a_0 b_0 + a_1 b_1 + \ldots + a_n b_n$. (Note the similarity in form to the dot product.)

- If $\mathbf{p} = a_0 + a_1 x + \ldots + a_n x^n$ and $\mathbf{q} = b_0 + b_1 x + \ldots + b_n x^n$ are polynomials in P_n and if $x_0, x_1, \ldots, x_n$ are distinct real numbers, then the formula $\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)\, q(x_0) + p(x_1)\, q(x_1) + \ldots + p(x_n)\, q(x_n)$ is the **evaluation inner product** at $x_0, x_1, \ldots, x_n$.

- If $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two functions in $C[a, b]$, then $\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x) g(x) dx$ defines an **inner product** on $C[a, b]$.

- If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n expressed in column form, A is an invertible $n \times n$ matrix, and $\mathbf{u} \cdot \mathbf{v}$ is the Euclidean inner product on R^n, then the formula $\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}$ is the **inner product on $\mathbf{R^n}$ generated by A**. This is an example of a **matrix inner product**.

#10 Compute the standard inner product on M_{22} of the given matrices.

$U = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}$, $V = \begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix}$

$$\langle \vec{u}, \vec{v} \rangle = tr(U^T V)$$

$$U^T = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix} \Rightarrow U^T V = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & -18 \\ 8 & 52 \end{bmatrix}$$

$$tr(U^T V) = 4 + 52 = 56$$

Consider $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$, $V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$

$$U^T V = \begin{bmatrix} u_1 & u_3 \\ u_2 & u_4 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} = \begin{bmatrix} u_1 v_1 + u_3 v_3 & u_1 v_2 + u_3 v_4 \\ u_2 v_1 + u_4 v_3 & u_2 v_2 + u_4 v_4 \end{bmatrix}$$

$$tr(U^T V) = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

(Basically a dot product)

#11 Find the standard inner product on P_2 of the given polynomials.

$\mathbf{p} = -2 + x + 3x^2$, $\mathbf{q} = 4 - 7x^2$

$$\langle \vec{p}, \vec{q} \rangle = -2(4) + 1(0) + 3(-7)$$

$$= -29$$

#16 A sequence of sample points is given. Use the evaluation inner product on P_3 at those sample points to find $\langle \mathbf{p}, \mathbf{q} \rangle$ for the polynomials $\mathbf{p} = x + x^3$ and $\mathbf{q} = 1 + x^2$.
$x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$

x_i	$p(x_i)$	$q(x_i)$
-1	-2	2
0	0	1
1	2	2
2	10	5

$$\langle \vec{p}, \vec{q} \rangle = -2(2) + 0(1) + 2(2) + 10(5) = 50$$

#8 Use the inner product on R^2 generated by the matrix A to find $\langle \mathbf{u}, \mathbf{v} \rangle$ for the vectors
$\mathbf{u} = (0, -3)$ and $\mathbf{v} = (6, 2)$.

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$\langle \vec{u}, \vec{v} \rangle = A\vec{u} \cdot A\vec{v}$$

$$A\vec{u} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ -9 \end{bmatrix}, \quad A\vec{v} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \end{bmatrix}$$

$$A\vec{u} \cdot A\vec{v} = (-3, -9) \cdot (14, 0) = -42$$

Since $\vec{u} \cdot \vec{v} = \vec{v}^T \vec{u}$ (Table 1 in 3.2), we have

$$\langle \vec{u}, \vec{v} \rangle = (A\vec{v})^T A\vec{u} = \vec{v}^T A^T A \vec{u}.$$

Example: A weighted Euclidean inner product

For this class we have HW: 10%, Exams: 60%, Final exam: 30%. Suppose we have 4 exams, each worth 100 pts. The final has 100 pts possible, and total HW is 320 pts.

Scores for Student X | Total possible points

HW: 265 | 320

Exams: 350 | 400

Final: 85 | 100

$\vec{u} = (265, 350, 85)$ is points earned

$\vec{v} = \left(\frac{1}{320}, \frac{1}{400}, \frac{1}{100}\right)$ is base ~~for~~ each category

$\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v} = \frac{265}{320} + \frac{350}{400} + \frac{85}{100}$

does not factor category weights

Define $\langle \vec{u}, \vec{v} \rangle = \underline{10}\left(\frac{265}{320}\right) + \underline{60}\left(\frac{350}{400}\right) + \underline{30}\left(\frac{85}{100}\right)$

≈ 86

This is a weighted Euclidean inner product with weights $w_1 = 10$, $w_2 = 60$, $w_3 = 30$.

Considering a matrix inner product, a weighted Euclidean inner product is generated by

$A = \begin{bmatrix} \sqrt{w_1} & & & 0 \\ & \sqrt{w_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{w_n} \end{bmatrix}$ so that $A^T A = \begin{bmatrix} w_1 & & & 0 \\ & w_2 & & \\ & & \ddots & \\ 0 & & & w_n \end{bmatrix}$

Definition: If V is a real inner product space, then the **norm** or **length** of a vector $\mathbf{v}$ in V is denoted by $\|\mathbf{v}\|$ and is defined by $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ and the **distance** between two vectors is denoted by $d(\mathbf{u}, \mathbf{v})$ and is defined by $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$. A vector of norm 1 is called a **unit vector**.

#1 Let R^2 have the weighted Euclidean inner product $\langle \mathbf{u},\ \mathbf{v}\rangle = 2u_1v_1 + 3u_2v_2$ and let $\mathbf{u} = (1,\ 1)$, $\mathbf{v} = (3,\ 2)$, $\mathbf{w} = (0,\ -1)$, and $k = 3$. Compute the stated quantities.
a. $\langle \mathbf{u},\ \mathbf{v}\rangle$ b. $\langle k\mathbf{u},\ \mathbf{v}\rangle$ c. $\langle \mathbf{u}+\mathbf{v},\ \mathbf{w}\rangle$ d. $\|\mathbf{v}\|$ e. $d(\mathbf{u},\ \mathbf{v})$ f. $\|\mathbf{u} - k\mathbf{v}\|$

a) $\langle \vec{u}\cdot\vec{v}\rangle = 2(1)(3) + 3(1)(2) = 12$

b) $\langle 3\vec{u},\vec{v}\rangle = 2(3)(3) + 3(3)(2) = 36 = 3\langle \vec{u},\vec{v}\rangle$

c) $\vec{u}+\vec{v} = \langle 4, 3\rangle \Rightarrow \langle \vec{u}+\vec{v},\vec{w}\rangle = 2(4)(0) + 3(3)(-1)$

$\langle \vec{u},\vec{w}\rangle = 2(1)(0) + 3(1)(-1) = -3$ $= -9$

$\langle \vec{v},\vec{w}\rangle = 2(3)(0) + 3(2)(-1) = -6$ $\langle \vec{u}+\vec{v},\vec{w}\rangle = \langle \vec{u},\vec{w}\rangle + \langle \vec{v},\vec{w}\rangle$

d) $\|\vec{v}\| = \sqrt{\langle \vec{v},\vec{v}\rangle} = \sqrt{2(3)^2+3(2)^2} = \sqrt{30}$

e) $d(\vec{u},\vec{v}) = \sqrt{\langle \vec{u}-\vec{v}, \vec{u}-\vec{v}\rangle} = \sqrt{2(-2)^2+3(-1)^2}$

$\vec{u}-\vec{v} = \langle -2, -1\rangle$ $= \sqrt{11}$

Theorem 6.1.1 Norms and Distances in Inner Product Spaces
If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a real inner product space V, and if k is a scalar, then:
a) $\|\mathbf{v}\| \geq 0$ with equality if and only if $\mathbf{v} = \mathbf{0}$ (analogous to Theorem 3.2.1 (a) and (b)).
b) $\|k\mathbf{v}\| = |k|\,\|\mathbf{v}\|$ (analogous to Theorem 3.2.1 (c)).
c) $d(\mathbf{u},\ \mathbf{v}) = d(\mathbf{v},\ \mathbf{u})$ (analogous to Theorem 3.2.2 (d)).
d) $d(\mathbf{u},\ \mathbf{v}) \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{v}$ (ibid).

#37 Let the vector space P_2 have the inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx$. Find the following for $\mathbf{p} = 1$ and $\mathbf{q} = x^2$.

a. $\langle \mathbf{p}, \mathbf{q} \rangle$ b. $d(\mathbf{p}, \mathbf{q})$ c. $\|\mathbf{p}\|$ d. $\|\mathbf{q}\|$

a) $\langle \vec{p}, \vec{q} \rangle = \int_{-1}^{1} x^2 dx = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$

b) $d(\vec{p}, \vec{q}) = \sqrt{\langle \vec{p}-\vec{q}, \vec{p}-\vec{q} \rangle} = \left(\int_{-1}^{1} (1-x^2)^2 dx\right)^{1/2}$

$= 2\int_{\cancel{-1}\ 0}^{1} (1-2x^2+x^4)dx = \frac{4}{\sqrt{15}}$

c) $\|\vec{p}\| = \sqrt{\langle \vec{p}, \vec{p} \rangle} = \left(\int_{-1}^{1} 1 dx\right)^{1/2} = \sqrt{2}$

d) $\|\vec{q}\| = \left(\int_{-1}^{1} x^4 dx\right)^{1/2} = \sqrt{\frac{2}{5}}$

Definition: If V is an inner product space, then the set of points in V that satisfy $\|\mathbf{u}\| = 1$ is called the **unit sphere** in V (or the **unit circle** in the case where $V = R^2$).

The Euclidean inner product on $\mathbb{R}^2$ gives the unit circle.

A weighted Euclidean inner product gives a distorted circle, a.k.a. an ellipse.

In the case of the integral inner product like in #37, the unit sphere consists of all functions $\vec{f} \in C[a,b]$ such that

$$\int_a^b [f(x)]^2\,dx = 1$$

Ex: $\int_{-1}^{1} \frac{1}{2}\,dx = 1 \Rightarrow f(x) = \frac{1}{\sqrt{2}}$

$$\int_0^{\pi} \sin x\,dx = 1 \Rightarrow f(x) = \sqrt{\sin x}$$

Theorem 6.1.2 Algebraic Properties of Inner Products (generalization of Theorem 3.2.3)

If **u, v** and **w** are vectors in a real inner product space V, and if k is a scalar, then:

a) $\langle \mathbf{0},\ \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{0}\rangle = 0$

b) $\langle \mathbf{u},\ \mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{v}\rangle + \langle \mathbf{u},\ \mathbf{w}\rangle$

c) $\langle \mathbf{u},\ \mathbf{v}-\mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{v}\rangle - \langle \mathbf{u},\ \mathbf{w}\rangle$

d) $\langle \mathbf{u}-\mathbf{v},\ \mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{w}\rangle - \langle \mathbf{v},\ \mathbf{w}\rangle$

e) $k\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u},\ k\mathbf{v}\rangle$

proving these is a good exercise.